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On Dirac fields in a curved space–time

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Abstract. It is shown that spin-$\frac{1}{2}$ particles with non-zero rest-mass have certain anomalous properties in a curved space–time. The sign of their energy density may be observer-dependent. In addition, the complete energy–momentum tensor may vanish in some space–times.

1. Introduction

It is known that the two-component neutrino field has a number of anomalous properties when considered in a curved space–time. For example, the sign of its energy density is observer-dependent. In fact the sign of the energy density of a neutrino field may even change as it moves through space. In addition, as the neutrino propagates, its complete energy–momentum tensor may suddenly vanish for a period even though its current vector remains non-zero. A non-trivial solution with zero energy–momentum tensor is known as a ghost neutrino.

I have pointed out in recent articles (Griffiths 1977, 1979) that these anomalous properties of neutrino fields are inevitable. They are a consequence of the fact that quantities describing the local curvature of space–time enter explicitly into the energy–momentum tensor. The expression for the neutrino energy–momentum tensor contains derivatives of spinor quantities. In a curved space–time ordinary derivatives are replaced by covariant derivatives which include terms describing the local curvature. Now, since curvature terms may be positive or negative and depend on the observer, it is not surprising that even the sign of the energy density is observer-dependent. Ghost solutions correspond to the particular case in which the curvature terms in the energy–momentum tensor actually cancel out the terms involving the partial derivatives.

It has sometimes been argued that the anomalies found in neutrino fields are peculiar to two-component neutrinos and that if neutrinos are considered to have non-zero mass the anomalies would not occur. In this paper I show that this suggestion is incorrect. If we take a four-component Dirac field with non-zero mass, then in a curved space–time the anomalies both of energy non-definiteness and of ghost solutions still occur. They inevitably arise because curvature terms enter the energy–momentum tensor of the Dirac field.

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2. Field equations

The Dirac field is usually described by a four-component spinor. However I find it convenient here to use two two-component spinors ψ^A and ϕ^A as described by Corson (1953). Spinor indices, denoted here by capital latin letters, take the values 1 and 2.

In order to describe the field in a curved space–time, it is necessary first to generalise the complex Pauli spin matrices so that they satisfy

$$\sigma^{\mu}_{A\dot{B}}\sigma^{\nu C\dot{B}} + \sigma^{\nu}_{A\dot{B}}\sigma^{\mu C\dot{B}} = 2g^{\mu\nu}\delta^{C}_{A}$$

at all points of space–time for which the metric is $g^{\mu\nu}$. It is also necessary to replace partial derivatives by covariant derivatives.

Having made these two generalisations, Dirac's equation for a spin-$\frac{1}{2}$ particle can be written in the form:

$$\sigma^{\mu\dot{A}B}\psi_{B;\mu} = m\phi^{\dot{A}}$$

$$\sigma^{\mu}_{A\dot{B}}\phi^{\dot{B}}_{;\mu} = -m\psi_A.$$

The probability current vector is given by

$$j_\mu = \sigma_{\mu A\dot{B}}(\psi^A\psi^{\dot{B}} + \phi^A\phi^{\dot{B}})$$

and the energy–momentum tensor of the field is

$$E_{\mu\nu} = \tfrac{1}{4}\mathrm{i}[\sigma_{\mu A\dot{B}}(\psi^A\psi^{\dot{B}}_{;\nu} - \psi^{\dot{B}}\psi^{A}_{;\nu} - \phi^A\phi^{\dot{B}}_{;\nu} + \phi^{\dot{B}}\phi^{A}_{;\nu})$$
$$- \sigma_{\nu A\dot{B}}(\psi^A\psi^{\dot{B}}_{;\mu} - \psi^{\dot{B}}\psi^{A}_{;\mu} - \phi^A\phi^{\dot{B}}_{;\mu} + \phi^{\dot{B}}\phi^{A}_{;\mu})].$$

It is now convenient to introduce the notation of Newman and Penrose (1962). We introduce two basis spinors o^A and ι^A, normalised such that

$$o_A\iota^A = -\iota_A o^A = 1.$$

These can be used to define a tetrad of null vectors l_μ, n_μ, m_μ, $\bar{m}_\mu$ by putting

$$l_\mu = \sigma_{\mu A\dot{B}}o^A o^{\dot{B}}, \qquad n_\mu = \sigma_{\mu A\dot{B}}\iota^A\iota^{\dot{B}}, \qquad m_\mu = \sigma_{\mu A\dot{B}}o^A\iota^{\dot{B}}.$$

The spin coefficients can be defined as the components of the covariant derivatives of the basis spinors as follows:

$$o_{A;\mu}\sigma^{\mu}_{B\dot{C}} = \gamma o_A o_B o_{\dot{C}} - \alpha o_A o_B \iota_{\dot{C}} - \beta o_A \iota_B o_{\dot{C}} + \epsilon o_A \iota_B \iota_{\dot{C}}$$
$$- \tau\iota_A o_B o_{\dot{C}} + \rho\iota_A o_B \iota_{\dot{C}} + \sigma\iota_A\iota_B o_{\dot{C}} - \kappa\iota_A\iota_B\iota_{\dot{C}}$$

$$\iota_{A;\mu}\sigma^{\mu}_{B\dot{C}} = \nu o_A o_B o_{\dot{C}} - \lambda o_A o_B \iota_{\dot{C}} - \mu o_A \iota_B o_{\dot{C}} + \pi o_A \iota_B \iota_{\dot{C}}$$
$$- \gamma\iota_A o_B o_{\dot{C}} + \alpha\iota_A o_B \iota_{\dot{C}} + \beta\iota_A\iota_B o_{\dot{C}} - \epsilon\iota_A\iota_B\iota_{\dot{C}}.$$

We now assume that the two spinors of the Dirac field are not proportional to each other. (The exceptional case in which this is not so is treated separately in § 5.) In this general case it is possible to align the two basis spinors with the field spinors. We can put $\psi^A = \psi o^A$ and $\phi^A = \phi\iota^A$ where ψ and ϕ are complex scalars. This choice does not define the basis spinors uniquely. We can still make use of the freedom

$$o^A \to R\,\mathrm{e}^{\mathrm{i}S}o^A, \qquad \iota^A \to R^{-1}\,\mathrm{e}^{-\mathrm{i}S}\iota^A.$$

Using the notation of Newman and Penrose (1962), the Dirac equation can now be written in the form:

$$D\psi = (\rho - \epsilon)\psi + m\bar{\phi}$$

$$\delta\psi = (\tau - \beta)\psi$$

$$\bar{\delta}\phi = -(\pi - \alpha)\phi$$

$$\Delta\phi = -(\mu - \gamma)\phi - m\bar{\psi}.$$

The current vector is now

$$j_\mu = \psi\bar{\psi}l_\mu + \phi\bar{\phi}n_\mu$$

and the energy–momentum tensor can be written in terms of its tetrad components as

$$E_{\mu\nu} = Al_\mu l_\nu + 2Bl_{(\mu}m_{\nu)} + 2\bar{B}l_{(\mu}\bar{m}_{\nu)} + 2Cl_{(\mu}n_{\nu)} + 2Gm_{(\mu}\bar{m}_{\nu)} + Dm_\mu m_\nu + \bar{D}\bar{m}_\mu\bar{m}_\nu + 2En_{(\mu}m_{\nu)} + 2\bar{E}n_{(\mu}\bar{m}_{\nu)} + Fn_\mu n_\nu$$

where

$$A = \tfrac{1}{2}\mathrm{i}[\psi\Delta\bar{\psi} - \bar{\psi}\Delta\psi + \psi\bar{\psi}(\bar{\gamma} - \gamma)]$$

$$B = \tfrac{1}{4}\mathrm{i}[\bar{\psi}\bar{\delta}\psi + \psi\bar{\psi}(\alpha - 2\bar{\tau}) + \phi\bar{\phi}\nu]$$

$$C = \tfrac{1}{4}\mathrm{i}[\psi\bar{\psi}(\bar{\rho} - \rho) + \phi\bar{\phi}(\bar{\mu} - \mu) + 2m\psi\phi - 2m\bar{\psi}\bar{\phi}]$$

$$G = \tfrac{1}{4}\mathrm{i}[\psi\bar{\psi}(\bar{\rho} - \rho) + \phi\bar{\phi}(\bar{\mu} - \mu)]$$

$$D = \tfrac{1}{2}\mathrm{i}[\psi\bar{\psi}\bar{\sigma} - \phi\bar{\phi}\lambda]$$

$$E = \tfrac{1}{4}\mathrm{i}[\phi\bar{\delta}\bar{\phi} + \phi\bar{\phi}(2\pi - \bar{\beta}) - \psi\bar{\psi}\bar{\kappa}]$$

$$F = \tfrac{1}{2}\mathrm{i}[\bar{\phi}D\phi - \phi D\bar{\phi} + \phi\bar{\phi}(\bar{\epsilon} - \epsilon)].$$

These components have been simplified as far as possible using the components of the Dirac equation.

3. Energy conditions

Following Wainwright (1971) we introduce two energy conditions to apply to the field:

E_1: A field is said to be of type E_1 if its energy–momentum tensor satisfies $E_{\mu\nu}u^\mu u^\nu \neq 0$ for all (unit, future pointing) time-like vectors u^μ at each event for which $E_{\mu\nu} \neq 0$.

E_2: A field is said to be of type E_2 if its energy–momentum tensor is such that $E_{\mu\nu}u^\nu$ is a time-like or null vector for all time-like vectors u^μ at each event for which $E_{\mu\nu} \neq 0$.

The condition E_1 is equivalent to the condition that the field should have either positive energy density or negative energy density with respect to all observers. The condition E_2 is the condition that the four-momentum of the field is time-like or null with respect to all observers. Both these conditions are eminently reasonable and are usually automatically satisfied for any realistic field. However they are not satisfied for a neutrino field. It is the purpose of this section to show that they are not automatically satisfied for the Dirac field either, even though the field has a non-zero mass parameter.

The results are stated here in two theorems which relate to the components of the energy–momentum tensor defined in the previous section.

Theorem 1. The Dirac field is of type E_1 if and only if there exist no real roots of the quartic

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$$

where

$$\begin{aligned}
a_0&=A\\
a_1&=2h(B\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{B}\,\mathrm{e}^{\mathrm{i}\theta})\\
a_2&=2C+2h^2G+h^2(D\,\mathrm{e}^{-2\mathrm{i}\theta}+\bar{D}\,\mathrm{e}^{2\mathrm{i}\theta})\\
a_3&=2h(E\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{E}\,\mathrm{e}^{\mathrm{i}\theta})\\
a_4&=F
\end{aligned}$$

for all values of θ and for all values of h satisfying $0\leqslant h<1$.

Proof. Consider the vector

$$u_\mu=al_\mu+bn_\mu-c\,\mathrm{e}^{\mathrm{i}\theta}m_\mu-c\,\mathrm{e}^{-\mathrm{i}\theta}\bar{m}_\mu$$

which is a general time-like vector provided $ab>c^2$. Then

$$\begin{aligned}
E_{\mu\nu}u^\mu u^\nu={}&b^2A+2bc(B\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{B}\,\mathrm{e}^{\mathrm{i}\theta})+2abC+2c^2G\\
&+c^2(D\,\mathrm{e}^{-2\mathrm{i}\theta}+\bar{D}\,\mathrm{e}^{2\mathrm{i}\theta})+2ac(E\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{E}\,\mathrm{e}^{\mathrm{i}\theta})+a^2F.
\end{aligned}$$

The field clearly satisfies condition E_1 if and only if this expression has no real roots for a, b and c satisfying $ab>c^2$ for any value of θ. Now putting $c^2=h^2ab$ where $0\leqslant h<1$ and dividing by b^2, the above expression becomes (on putting $x^2=a/b$)

$$\begin{aligned}
&A+2h(B\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{B}\,\mathrm{e}^{\mathrm{i}\theta})x+[2C+2h^2G+h^2(D\,\mathrm{e}^{-2\mathrm{i}\theta}+\bar{D}\,\mathrm{e}^{2\mathrm{i}\theta})]x^2\\
&\qquad+2h(E\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{E}\,\mathrm{e}^{\mathrm{i}\theta})x^3+Fx^4.
\end{aligned}$$

The freedom in the choice of θ permits us to consider negative values of x as well as positive values and the theorem follows.

Theorem 2. The Dirac field is of type E_2 if and only if

$$b_0+b_1x+b_2x^2+b_3x^3+b_4x^4\geqslant 0$$

for all x, where:

$$\begin{aligned}
b_0&=AC-B\bar{B}\\
b_1&=h[A(E\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{E}\,\mathrm{e}^{\mathrm{i}\theta})+(C-G)(B\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{B}\,\mathrm{e}^{\mathrm{i}\theta})-(B\bar{D}\,\mathrm{e}^{\mathrm{i}\theta}+\bar{B}D\,\mathrm{e}^{-\mathrm{i}\theta})]\\
b_2&=C^2+AF-B\bar{E}-\bar{B}E-h^2(B\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{B}\,\mathrm{e}^{\mathrm{i}\theta})(E\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{E}\,\mathrm{e}^{\mathrm{i}\theta})\\
&\qquad-h^2(G+D\,\mathrm{e}^{-2\mathrm{i}\theta})(G+\bar{D}\,\mathrm{e}^{2\mathrm{i}\theta})\\
b_3&=h[F(B\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{B}\,\mathrm{e}^{\mathrm{i}\theta})+(C-G)(E\,\mathrm{e}^{-\mathrm{i}\theta}+\bar{E}\,\mathrm{e}^{\mathrm{i}\theta})-(E\bar{D}\,\mathrm{e}^{\mathrm{i}\theta}+\bar{E}D\,\mathrm{e}^{-\mathrm{i}\theta})]\\
b_4&=CF-E\bar{E}
\end{aligned}$$

for all values of θ and for all values of h satisfying $0\leqslant h<1$.

Proof. Using the notation of the previous proof we consider the vector

$$u_\mu = b(x^2 l_\mu + n_\mu - hx\,e^{i\theta} m_\mu - hx\,e^{-i\theta} \bar{m}_\mu)$$

which is time-like for all values of x, θ and h satisfying $0 \leqslant h < 1$. The condition E_2 is the condition

$$E_{\alpha\mu} E^{\alpha}_{\nu} u^\mu u^\nu \geqslant 0.$$

Direct substitution yields the theorem.

The conditions described in these theorems represent very severe restrictions on the components of the energy–momentum tensor. They will not be satisfied in general and exact solutions will exist which are not of types E_1 or E_2. In such solutions even the sign of the energy density is observer-dependent. Such anomalous situations arise because terms describing the local curvature of space–time enter explicitly into the energy–momentum tensor.

It is also worth pointing out that the condition E_2 does not in general imply E_1 for a Dirac field as it does for a two-component neutrino field.

4. Ghost solutions

As well as the anomaly of energy non-definiteness we must also consider the anomaly of the existence of ghost solutions. In such solutions the complete energy–momentum tensor vanishes although the current vector is non-zero. In flat space–times such solutions are trivial: they correspond to the field spinors being constants. However in a curved space–time they are non-trivial: they correspond to the situation in which the terms describing the local curvature of space–time that appear in the energy–momentum tensor actually cancel out the terms involving the partial derivatives.

To obtain ghost solutions we solve the Dirac equation and Einstein's vacuum gravitational field equations subject to the condition that all the components of the energy–momentum tensor are zero. It is possible to use the freedom in the choice of basis spinor to equate the two scalar components ψ and ϕ, and the condition that C and G are zero then implies that these are real ($\phi = \psi = \bar{\psi}$). The components of the Dirac equation then become:

$$D\psi = (m + \rho - \epsilon)\psi$$

$$\delta\psi = (\tau - \beta)\psi$$

$$\Delta\psi = -(m + \mu - \gamma)\psi$$

which together with the condition that $E_{\mu\nu} = 0$ imply that

$$\rho = \bar{\rho}, \qquad \epsilon = \bar{\epsilon}, \qquad \gamma = \bar{\gamma}, \qquad \mu = \bar{\mu}, \qquad \lambda = \bar{\sigma},$$

$$\pi = \alpha + \bar{\beta} - \bar{\tau}, \qquad \nu = -\alpha + \bar{\beta} + \bar{\tau}, \qquad \kappa = 2\bar{\alpha} - \tau.$$

It is now very difficult to proceed any further in the general case. We have exhausted the freedom in the choice of tetrad, but the field equations are not yet in a form that is easily integrable. I will therefore merely demonstrate the existence of ghost neutrinos by giving a particular example. This has been obtained using Newman–Penrose techniques. Detailed calculations are omitted here.

It is difficult to integrate the Newman–Penrose equations unless the space–time possesses a shear-free congruence of null geodesics in which case it is algebraically special. It is therefore convenient to introduce the assumptions that $\kappa = 0$ and $\sigma = 0$. It is then possible to show that the field equations are inconsistent unless in addition $\alpha = 0$ and $\beta = 0$. The gravitational field equations in this case yield the simple type D metric

$$ds^2 = 2(au+bv)^{-1/2}\,du\,dv - (au+bv)(dx^2+dy^2)$$

where u and v are two null coordinates and a and b are arbitrary constants. The components of the Dirac equation become:

$$\partial/\partial v[\lg\psi + \tfrac{3}{8}\lg(au+bv) + \tfrac{1}{2}X] = m(au+bv)^{-1/4}\,e^{-X}$$

$$\partial/\partial u[\lg\psi + \tfrac{3}{8}\lg(au+bv) - \tfrac{1}{2}X] = -m(au+bv)^{-1/4}\,e^{X}$$

where $\psi = \psi(u, v)$, and $X = X(u, v)$ satisfies the equation

$$\frac{\partial^2 X}{\partial u\,\partial v} + m(au+bv)^{-1/4}\left[e^{X}\left(\frac{b}{4}(au+bv)^{-1} - \frac{\partial X}{\partial v}\right) + e^{-X}\left(\frac{a}{4}(au+bv)^{-1} + \frac{\partial X}{\partial u}\right)\right] = 0.$$

It is clear that solutions of these equations exist. The latter equation is the integrability condition for the previous two. However the meaning of the solution is unclear, except that it is the Dirac field whose energy–momentum tensor vanishes in the given space–time.

5. The null-field case

We now return to the case in which the two field spinors are aligned. In this case the current vector is null and the energy–momentum tensor is trace-free. We may align one of the basis spinors with the field spinors and put

$$\psi^A = \psi o^A, \qquad \phi^A = \phi o^A.$$

The basis spinors are now defined up to the transformation

$$o^A \to R\,e^{iS} o^A, \qquad \iota^A \to R^{-1}\,e^{-iS}\iota^A + T o^A$$

where T is complex. The Dirac equation can now be written in the form

$$D\psi = (\rho - \epsilon)\psi, \qquad D\phi = (\rho - \epsilon)\phi,$$

$$\delta\psi = (\tau - \beta)\psi - m\bar{\phi}, \qquad \delta\phi = (\tau - \beta)\phi - m\bar{\psi},$$

and the components of the energy–momentum tensor are:

$$A = \tfrac{1}{2}i[\psi\Delta\bar{\psi} - \bar{\psi}\Delta\psi - \phi\Delta\bar{\phi} + \bar{\phi}\Delta\phi + (\psi\bar{\psi} - \phi\bar{\phi})(\bar{\gamma} - \gamma)]$$

$$B = \tfrac{1}{4}i[\bar{\psi}\bar{\delta}\psi - \bar{\phi}\bar{\delta}\phi + (\psi\bar{\psi} - \phi\bar{\phi})(\alpha - 2\bar{\tau})]$$

$$C = G = \tfrac{1}{4}i(\psi\bar{\psi} - \phi\bar{\phi})(\bar{\rho} - \rho)$$

$$D = \tfrac{1}{2}i(\psi\bar{\psi} - \phi\bar{\phi})\bar{\sigma}$$

$$E = -\tfrac{1}{4}i(\psi\bar{\psi} - \phi\bar{\phi})\bar{\kappa}$$

$$F = 0.$$

The theorems relating to energy conditions can be applied to this case directly. We find that

$$\mathrm{E}_1 \Rightarrow \begin{cases} E=0 \\ AC-B\bar{B}\geqslant 0, \\ 2|C|\geqslant D \end{cases} \qquad \mathrm{E}_2 \Rightarrow \begin{cases} E=0 \\ AC-B\bar{B}\geqslant 0 \\ D=0 \end{cases}$$

Thus unless $\psi\bar{\psi}=\phi\bar{\phi}$, a null Dirac field can be of type E_2 only if its current vector is aligned with a shear-free geodesic null congruence. Clearly energy non-definite solutions exist in this case.

In the trivial case in which ψ and ϕ are equal the energy–momentum tensor vanishes. However there also exist ghost solutions for which ϕ and ψ are not equal.

6. Conclusions

It has been shown that the anomalies associated with neutrino fields in general relativity also occur with Dirac fields. Such fields may be energy non-definite and ghost solutions may occur. For spin-$\frac{1}{2}$ particles with non-zero rest-mass, the sign of the energy density is generally dependent upon the observer. Thus spin-$\frac{1}{2}$ particles do not in general satisfy the energy condition that is required for the singularity theorems in general relativity (Hawking and Ellis 1973). For weak gravitational fields the anomalies considered here may never occur. However in a situation of gravitational collapse it is not reasonable to expect Dirac particles to satisfy the usual energy conditions. It would therefore be interesting to see if singularity theorems can be proved under weaker assumptions of energy density.

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